# Analytic model of near-field radio-frequency sheaths:

# II. Full plasma dielectric

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# Analytic model of near-field radio-frequency sheaths:
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## Abstract

An analytic model is derived for electromagnetic radio-frequency (rf) wave propagation in a plasma-filled waveguide with rf sheath boundary conditions. The model gives a simplified description of the rf fields and sheath potentials near an ion cyclotron range of frequencies (ICRF) antenna under certain conditions. The present work lifts the restriction to a low density plasma ("tenuous plasma model") described in a previous paper [D.A. D'Ippolito and J.R. Myra, Phys. Plasmas **16**, 022506 (2009)]  to include the full plasma dielectric tensor with the ordering $\varepsilon_\perp \sim \varepsilon_\times \sim 1, \ \ \varepsilon_\parallel \gg 1$ for the case where the magnetic field is well aligned with the antenna.  It is shown that retaining $\varepsilon_\times \sim 1$ provides an additional drive term for the rf sheath.  This effect is shown to be negligible in most practical situations. suggesting that the tenuous plasma model does not miss any essential finite-density effects. The condition to recover the tenuous plasma result is derived. Expressions for the sheath voltage and sheath power dissipation are given in the arbitrary density limit, and a comparison of several mechanisms for dissipating power in rf sheaths is discussed.

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## I. Introduction

Radiofrequency (rf) waves in the ion cyclotron range of frequencies (ICRF) have been used successfully to heat and drive current in many fusion experiments and are one of the planned heating mechanisms in ITER.[1] An important issue for optimizing ICRF heating is controlling the strongly nonlinear interactions associated with the unwanted, but parasitically-coupled, slow wave. A review of the early history of observed antenna-plasma interactions in experiments is given in Ref. 2 and a review of important nonlinear mechanisms is given in Ref. 3. One of the most important interactions to be minimized during ICRF heating is the formation of *rf sheaths*.[4-8] The basic physics of rf sheath formation, and the importance of its control for successful rf heating and current drive, was described in Ref. 3.

The present paper is the second in a series of papers developing an analytic model of rf sheath formation on ICRF antennas. The first paper[9] (hereafter referred to as I) used the low-density or "tenuous plasma" limit ($\varepsilon_x = 0$, $\varepsilon_\perp = 1$, $|\varepsilon_\parallel| \gg 1$) to calculate antenna sheaths driven by magnetic field tilt; the present paper uses the full plasma dielectric tensor $\boldsymbol{\varepsilon}$ with the ordering $\varepsilon_\perp \sim \varepsilon_x \sim 1$, $\varepsilon_\parallel \gg 1$ to calculate the dominant finite-density effects on the sheaths (see Sec. II). This model allows us to calculate analytically some important properties of the coupling of the fast wave (FW) to the slow wave (SW) and the resulting antenna sheaths (e.g. the model describes the phasing, voltage and density dependence of the sheath potential). The results described here are useful for understanding the physics of near-field sheath formation and for illustrating the use of a sheath boundary condition[10,11] (SBC) proposed earlier. Work is in progress to incorporate this boundary condition (BC) into the TOPICA antenna code,[12] and the present model may also prove useful for future benchmarking of this code.

Although rf sheaths have been modeled[4-8,13-17] and their consequences studied experimentally[6,18-24] for over twenty years, the computational tools to make *quantitative*

predictions of rf sheath effects are not yet available. We have suggested a technique for carrying out quantitative calculations of the rf sheath potential in rf codes by means of a generalized boundary condition on material surfaces, which incorporates the sheath capacitance.[10,11] The present series of papers demonstrates the utility of this approach for calculations of antenna or near-field sheath potentials.

This work is part of a group of recent analytic and numerical calculations[9,11,25-27] carried out by the authors to illustrate the effects of the sheath BC in various physical situations. Taken together, these calculations provide insight into the dependence of sheath formation on the physical situation (driving wave, magnetic field geometry with respect to the boundary, etc.) A summary of these calculations is given in Appendix A for the interested specialist reader, to better put the present work into a more general context.

Here, it is sufficient to contrast the present calculation with that of paper I. Sheath formation on ICRF antennas that nominally launch fast waves involves two mechanisms for generating slow waves: (i) $\mathbf{J}_{\parallel}$ due to field line tilt, and (ii) FW-SW coupling by the axial BC. Both of these effects occur in the near field of the antenna (not to be confused with the far-field sheaths treated elsewhere.[10,25]) In case (i) the magnetic field is not perpendicular to the antenna current straps (i.e. $b_y \equiv B_y / B \neq 0$), and the antenna current $\mathbf{J}_a = \mathbf{J}_y$ has two components: $\mathbf{J}_{\perp} \sim \mathbf{J}_a$ and $\mathbf{J}_{\parallel} \sim b_y \mathbf{J}_a$. Here, (x, y, z) denote local coordinates in the radial, poloidal and toroidal directions, respectively, and the subscripts $\parallel$ and $\perp$ denote the components parallel and perpendicular to the local magnetic field line. For an ICRF antenna, the main $\mathbf{J}_{\perp}$ current drives the desired FW, and the small $\mathbf{J}_{\parallel}$ current drives a parasitic SW. Both waves have a $k_z$ given by the toroidal antenna structure and a $k_x$ that satisfies the appropriate (FW or SW) local dispersion relation (see Appendix B).

Most previous work on antenna sheaths has assumed that they are generated by mechanism (i), and the vacuum sheath model is used to estimate the sheath potential, i.e. $V_{sh} = \oint ds\, E_{\parallel}^{(vac)}$, where the integral is taken along the field line between sheaths, and $E_{\parallel}^{(vac)}$ is the vacuum rf electric field parallel to the equilibrium **B**. The resulting sheath potential is usually large (~ several hundred V) for field lines near the front face of high power ICRF antennas.[8,14,16-18,21] For an antenna immersed in a finite density plasma ($n_e > n_{LH}$, where $n_{LH}$ is the lower hybrid resonant density)  the *usual* SW due to mechanism (i) is driven at the antenna surface and is evanescent in front of the antenna with a short radial scale length, $L_x \sim c/\omega_{pe}$. (However, it is worth noting that under some circumstances the evanescent SW can couple to a propagating wave when sheath BC effects are taken into account.[27]) By "usual SW" we mean a wave on the SW branch which satisfies the ordering $n_{xs}^2 \sim \varepsilon_{\parallel}$. The sheath BC corrections to the usual SW are not discussed in paper I, but similar SW problems have been treated in earlier studies[11,26,27] (see Appendix A).

In contrast to this previous work, the calculations described here and in paper I are concerned with the new mechanism (ii), i.e. the SW is driven by the propagating FW as it interacts with the boundary sheaths. For this to occur requires the modified ordering $n_{xs}^2 \sim n_{xf}^2 \ll \varepsilon_{\parallel}$ (see below). The differences between the two papers is discussed further in Sec. II.  In both cases, the sheath BC couples the FW to a wave on the SW branch, but its wave vector satisfies unusual constraints, viz. $k_{xs} = k_x(FW)$ and $k_{zs}$ chosen to satisfy the sheath BC. This FW-driven SW can have a larger radial extent than the conventional one, because it is generated at each radial point by the FW as it propagates away from the antenna.  While the SW directly generated by misaligned current straps can be mitigated by (field-aligned) Faraday screens, this FW-generated SW is always present.

Assessing the relative importance of mechanisms (i) and (ii) for realistic antennas is outside the scope of this paper. For now, we make the following observations:

(a) When the field line is tilted, both (i) the conventional SW driven by $J_\parallel$ and (ii) the FW-driven SW (described in paper I) have amplitudes of the same order, viz. $b_y \hat{E}_y$, where $\hat{E}_y$ is the FW amplitude, when $n_x^2 (\Delta/L) \gg 1$.

(b) When the field line is not tilted, we will show in the present paper that the rf sheath is driven by anisotropy of the plasma dielectric ($\varepsilon_x \neq 0$); the resulting sheath is small, even at high density, and has no relation to the results of the vacuum sheath model.

(c) The SW in case (ii) has the same radial scale as the FW, potentially larger than that of the conventional SW driven by magnetic field line tilt and is not mitigated by aligning the magnetic field with the Faraday screens.

This paper is organized as follows. In Sec. II we discuss the antenna model and the justification for the various approximations used to simplify the calculations. In Sec. III we carry out the two-wave coupling calculation for the rf fields and sheath potential for the case $\varepsilon_x \neq 0$ and $b_y = 0$. It is shown that retaining $\varepsilon_x \sim 1$ provides an additional coupling term to the slow wave. In Sec. V, we evaluate several mechanisms for sheath power dissipation, which can be important at high density, and discuss the role of neutrals in providing collisional dissipation. A summary and conclusions are given in Sec. VI. Appendix A discusses the relation of the present work to a number of other rf sheath calculations carried out for different physical situations arising in a tokamak. Appendix B contains background material on the FW and SW dispersion relations and the ordering used in this paper.

## II. Antenna model

In paper I and here, we incorporate the sheath BC into a calculation of electromagnetic wave (coupled FW and SW) propagation in a plasma-filled waveguide.

The FW with amplitude $\hat{E}_y$ is launched at x = 0 and propagates in the +x direction. The same radial dependence ($\sim e^{ik_x x}$) is assumed for all waves, equivalent to an outgoing wave BC. The equilibrium magnetic field is given by $\mathbf{B} = B(\hat{e}_z + b_y \hat{e}_y)$ and the magnetic field lines intersect conducting walls at $z = \pm L$, where the wave satisfies the sheath BC described in Sec. III. To simplify the expressions, we set $k_y = 0$ and the field line tilt is assumed to be negligibly small, $b_y \equiv B_y / B << 1$, in contrast to paper I (see below). The boundaries at $z = \pm L$ represent the antenna frame or antenna protection limiters that enclose a typical FW antenna.

This model problem is meant to approximate the fields near the front face of a FW antenna, where the density is low but plasma effects can still be important. The FW electric field has a large component in the direction of the (poloidal) current straps. The situation where the magnetic field is perfectly aligned with respect to the antenna ($\mathbf{B} \cdot \mathbf{J}_a = 0$) almost never occurs in practice, but the misalignment is often small.

The assumption of constant density in the vicinity of the Faraday screen and antenna protection limiters is justified when the density gradient length is larger than that for the dc sheath potential. For example, in the limit of strong sheath voltages, rf convection[13] will flatten the density profile (in the radial direction) near the antenna. (This effect was demonstrated by reflectometry measurements of the local density profile in front of the antenna in Ref. 20.) This assumption also requires that poloidal inhomogeneities be weak, which is justified near the equatorial plane. A number of recent experimental papers have studied the spatial distribution of the density and sheath potential near the antenna and their radial penetration.[28,29-31] The constant density model provides a good test case for benchmarks and a simple limit in which to study the interaction of sheaths with plasma screening and wave propagation effects in the volume.

The assumption $k_y = 0$ on an infinite y domain is approximately valid near the poloidal midplane (y = 0) of a typical ICRF antenna and restricts the sheath drive to the

magnetic flux produced by the current straps (assuming a constant poloidal current distribution and field lines that pass in front of both current straps). It excludes treatment of a number of important effects, including sheaths in the corners of the antenna box[14] It also neglects such 3D effects as the magnetic flux of the current feeders and the effects of currents flowing in the Faraday screen and antenna box, which can be important in experiments.[24] All of these effects would require $k_y \neq 0$ and additional BCs in the y direction, but it is not possible to solve the problem analytically in such generality.

Finally, the assumption of definite parity in z for the launched rf wave is well satisfied at the antenna midplane (y = 0) for a typical two-strap ICRF antenna, with $E_{\parallel}$ having *even* (*odd*) parity in z corresponding to *monopole* (*dipole*) phasing. This type of antenna was common twenty years ago when ICRF heating was the main application, but has more recently been replaced by antennas with four or more current straps and non-symmetric phasings in order to have the capability for FW current drive. However, the two-strap symmetric case remains a good test problem for establishing the basic physics of antenna sheaths and providing test cases for benchmarking antenna codes, which are the main goals of the present calculation. A detailed discussion of the phasing dependence (monopole vs dipole) was given in I; in the present paper, we restrict the discussion of finite density effects to the monopole phasing case.

It is important to note that different orderings are used in papers I and II. In paper I, we solved the wave propagation problem in the tenuous plasma limit ($\varepsilon_x = 0$, $\varepsilon_{\perp} = 1$, $|\varepsilon_{\parallel}| \gg 1$) using a perturbation expansion in the small parameter $b_y$ and assuming a definite parity in z for the launched FW. Thus, finite-plasma effects entered only through $\varepsilon_{\parallel}$ and the field line tilt was the main effect driving the sheath. Here, we investigate a different coupling mechanism, assuming no field line tilt ($b_y = 0$) but retaining arbitrary density effects in the ion plasma dielectric response ($\varepsilon_x \neq 0$) using the ordering $\varepsilon_{\perp} \sim \varepsilon_x \sim 1$, $\varepsilon_{\parallel} \gg 1$. This problem couples two waves: a FW and SW of the

same parity. As in paper I, the SW must satisfy a special **k** ordering, and its amplitude is chosen so that the sum of the two waves satisfies the sheath BC.

## III. Basic equations

In this section, we discuss the formulation of the problem in which a FW and SW are coupled by the rf sheath BC in the absence of field line tilt.

### A. Wave physics

The wave equation for the rf electric field is

$$\mathbf{L}\mathbf{E} = -(4\pi i / \omega)\mathbf{J}_a \tag{1}$$

with the wave propagation operator defined as

$$\mathbf{L} = -\frac{c^2}{\omega^2}\nabla\times\left(\nabla\times\mathbf{E}\right) + \boldsymbol{\varepsilon}\cdot\mathbf{E} = \mathbf{n}\times(\mathbf{n}\times\mathbf{E}) + \boldsymbol{\varepsilon}\cdot\mathbf{E} = (\mathbf{nn} - n^2\mathbf{I} + \boldsymbol{\varepsilon})\cdot\mathbf{E} \ . \tag{2}$$

Here, **E** is the rf electric field, $\mathbf{J}_a$ is the antenna current density, $\mathbf{n} = \mathbf{k}\,c/\omega$ is the index of refraction, **k** is the wavenumber, $\omega$ is the rf frequency, **I** is the unit tensor and the plasma dielectric tensor is given by

$$\boldsymbol{\varepsilon} = \varepsilon_\perp\mathbf{I} + (\varepsilon_\parallel - \varepsilon_\perp)\mathbf{bb} + (i\varepsilon_\times/2)(\mathbf{b}\times\mathbf{I} + \mathbf{I}\times\mathbf{b}) \ , \tag{3}$$

where for ICRF waves we employ

$$\varepsilon_\perp = 1 - \frac{\omega_{pi}^2}{\left(\omega^2 - \Omega_i^2\right)} \ , \quad \varepsilon_x = \frac{\omega_{pi}^2\,\omega}{\Omega_i\left(\omega^2 - \Omega_i^2\right)} \ , \quad \varepsilon_\parallel = 1 - \frac{\omega_{pe}^2}{\omega^2} \ . \tag{4}$$

For a homogeneous plasma, the undriven problem ($\mathbf{L}\mathbf{E} = 0$) yields a fourth order dispersion relation in $n_x$ or $n_z$ for the coupled fast and slow wave roots. A more extensive discussion of the orderings used in this paper, the approximate dispersion relations, and their evaluation in the limit of small $1/\varepsilon_\parallel$ is given in Appendix B.

The wave equation becomes

$$LE_j = \begin{pmatrix} \varepsilon_\perp - n_z^2 & -i\varepsilon_\times & n_x n_z \\ i\varepsilon_\times & \varepsilon_\perp - n^2 & 0 \\ n_x n_z & 0 & \varepsilon_\parallel - n_x^2 \end{pmatrix} \begin{pmatrix} E_{xj} \\ E_{yj} \\ E_{zj} \end{pmatrix} = 0 \quad , \tag{5}$$

where $n^2 \equiv n_x^2 + n_z^2$ and the index $j = f,s$ identifies the wave (FW or SW). In the limit $\varepsilon_\parallel \gg n_x^2$, the third row gives the parallel polarization $\varepsilon_\parallel E_{zj} = -n_x n_z E_{xj}$, and the determinant of the remaining $2\times2$ system of equations gives the reduced dispersion relation

$$(\varepsilon_\perp - n^2)(\varepsilon_\perp - n_z^2) = \varepsilon_\times^2 \quad . \tag{6}$$

The two roots of this equation are the FW and the SW.

In the present calculation, these two rf fields are coupled by the sheath BC at each radial position, so we require that

$$n_{xf} = n_{xs} \equiv n_x \quad . \tag{7}$$

Thus, Eq. (6) is a quadratic equation for $n_z^2$. A useful identity relating both waves is given by

$$n_{zs}^2 - \varepsilon_\perp = \varepsilon_\perp - n_f^2 = \frac{\varepsilon_\times^2}{\varepsilon_\perp - n_{zf}^2} \quad . \tag{8}$$

If $\varepsilon_\times = 0$, Eq. (8) shows that the FW and SW dispersion relations reduce to $\varepsilon_\perp - n_f^2 = 0$ and $n_{zs}^2 - \varepsilon_\perp = 0$, respectively, as in paper I. For the general case, Eq. (8) shows that specifying $n_{zf}$ and the plasma dielectric is sufficient to determine $n_{zs}$. The dispersion relation and the derivation of Eq. (8) is discussed further in Appendix B.

For the monopole parity, the polarization of both waves can be written in the form

$$\mathbf{E}_j = A_j \left( iQ_j \cos k_{zj}z, \quad \cos k_{zj}z, \quad -\frac{Q_j}{F_j} \sin k_{zj}z \right) e^{ik_x x} e^{-i\omega t} \quad , \tag{9}$$

where $A_j$ are the wave amplitudes. We note the following identities for later use

$$\frac{Q_j}{F_j} = \frac{\varepsilon_\times}{\varepsilon_\parallel} G_j \quad , \tag{10}$$

$$\frac{-k_x \Delta \varepsilon_\parallel}{F_j} = n_x^2 \frac{\Delta}{L} \eta_{zj} \quad . \tag{11}$$

where $Q_j = \varepsilon_\times /(\varepsilon_\perp - n_{zj}^2)$, $F_j = -\varepsilon_\parallel /(n_x n_{zj})$, $G_j = n_x n_{zj} /(n_{zj}^2 - \varepsilon_\perp)$ and $\eta_{zj} \equiv k_{zj} L$, where 2L is the length of the system in the toroidal direction..

## B. Sheath BC

To complete the specification of the problem, the sheath BC[10,11]

$$\mathbf{E}_t = \nabla_t (\Delta \mathbf{D}_n) \tag{12}$$

is imposed at the sheath-plasma interfaces at $z = \pm L$:

$$E_x(-L) = \nabla_x [\Delta \varepsilon_\parallel E_z(-L)] \quad , \tag{13}$$

$$E_x(L) = -\nabla_x [\Delta \varepsilon_\parallel E_z(L)] \quad , \tag{14}$$

$$E_y(-L) = 0 \quad , \tag{15}$$

$$E_y(L) = 0 \quad , \tag{16}$$

where $k_y = 0$ was used in the BCs for $E_y$. This BC is derived using the continuity of the normal component of $\mathbf{D}$ (here, $D_z \equiv \varepsilon_{zz} E_z \approx \varepsilon_\parallel E_z$) and the tangential components of $\mathbf{E}$ across the sheath-plasma interface. The term on the rhs of the BC describes the effect of the sheath capacitance on the rf fields, where $\Delta$ is the sheath width in z. In the limit $\Delta \to 0$, we recover the usual "metal wall" BC, viz. that the tangential component of $\mathbf{E}$ vanishes. Here, "normal" (subscript n) and "tangential" (subscript t) are defined with respect to the sheath and wall.

For some solutions, the BCs at $z = \pm L$ yield identical constraints. Explicit calculation shows that this occurs in the two cases $b_y \neq 0$, $\varepsilon_x = 0$ and $b_y = 0$, $\varepsilon_x \neq 0$.

The general case (keeping both effects) does not have definite parity and requires the BCs at both ends.

Thus, we use the two BCs at $z = L$. Substituting the field solution from Eq. (9) into Eq. (14) yields the following relation between the wave amplitudes

$$\sum_{j=f,s} A_j \hat{X}_j \cos \eta_{zj} = 0 \tag{17}$$

where $\eta_{zj} \equiv k_{zj}L$ is related to the wavenumber $n_{zj}$ by $\eta_{zj} = n_{zj}\eta_0$ with $\eta_0 \equiv \omega L / c$. The coefficient $\hat{X}_j$ is defined by

$$\begin{aligned}\hat{X}_j &\equiv Q_j - k_x \Delta \varepsilon_\times G_j \tan \eta_{zj} \\ &= Q_j\left(1 + p\, \eta_{zj} \tan \eta_{zj}\right)\end{aligned} \tag{18}$$

where

$$p \equiv n_x^2(\Delta / L) \tag{19}$$

and the identities in Eq. (10) and (11) were used to obtain the form with parameter p. The BC in Eq. (16) yields the additional relation

$$\sum_{j=f,s} A_j \cos \eta_{zj} = 0 \quad . \tag{20}$$

These two equations, together with the two relations in Eq. (8), determine the coupled FW and SW amplitudes and the normalized wavenumbers $\eta_{zf}$ and $\eta_{zs}$. The two equations relating the wave amplitudes can be combined to give the global dispersion relation

$$1 = \frac{Q_s\left(1 + p\, \eta_{zs} \tan \eta_{zs}\right)}{Q_f\left(1 + p\, \eta_{zf} \tan \eta_{zf}\right)} \tag{21}$$

where

$$Q_f \equiv \frac{\varepsilon_\times}{\varepsilon_\perp - n_{zf}^2}, \quad Q_s \equiv \frac{\varepsilon_\times}{\varepsilon_\perp - n_{zs}^2} = \frac{\varepsilon_\perp - n_s^2}{\varepsilon_\times} \quad . \tag{22}$$

implying that

$$Q_f / Q_s \equiv \frac{\varepsilon_\times^2}{(\varepsilon_\perp - n_{zf}^2)(\varepsilon_\perp - n_s^2)} \quad . \tag{23}$$

It follows from Eq. (22) that $Q_f \to 0$ and $Q_s \to \infty$ in the tenuous plasma limit ($\varepsilon_x \to 0$). By Eq. (21) this limit requires either $1 + p\,\eta_{zs}\tan\eta_{zs} = 0$ or $1 + p\,\eta_{zf}\tan\eta_{zf} = \infty$  The latter condition implies that $\tan\eta_{zf} \to 0$ and $\eta_{zf} \to \pi/2$ for the lowest branch. (As in paper I, we consider here only the lowest mode of the system.)  We see that this calculation recovers the same solution for the normalized FW wave number $\eta_{zf}$ in the tenuous plasma limit as in paper I  but the sheath voltage will be different because the orderings are different.

## C.  Sheath voltage

Next, we derive an expression for the sheath voltage and show that it is caused by a new effect. To lowest (zero) order in $b_y$, the component of the rf electric field parallel to **B** is $E_\| \approx E_z$. Summing Eq. (9) over both waves and using Eq. (10), we obtain

$$\varepsilon_\| E_\| = -\varepsilon_\times \left( A_f G_f \sin k_{zf} z + A_s G_s \sin k_{zs} z \right) e^{ik_x x} e^{-i\omega t} \quad . \tag{24}$$

Note that the rhs is independent of $\varepsilon_\|$ so that $E_\| \propto 1/\varepsilon_\| \to 0$ as $\varepsilon_\| \to \infty$. This is the effect of plasma screening in our model.

We define the voltage across the plasma as in paper I, and find that

$$V_{pl} = \int_{-L}^{L} dz\, E_\| \to 0 \quad . \tag{25}$$

because $\varepsilon_\| E_\| \propto \sin kz$. The reason that the voltage drop across the plasma vanishes here is that we have eliminated the contribution to $E_\|$ proportional to $b_y$ which gave a non-zero result for $V_{pl}$ in paper I.  However, there is still a non-vanishing voltage drop across the sheaths, driven by a combination of the anisotropy of the plasma dielectric ($\varepsilon_\times \neq 0$) together with the sheath capacitance ($\Delta \neq 0$). The sheath voltage at $z = L$ is defined by

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$$V_{sh}(L) = -\int_{sh} dz\, E_{\parallel} = -\Delta\varepsilon_{\parallel}E_{\parallel}$$
$$= -\Delta\varepsilon_{\times}\left(A_f G_f \sin\eta_{zf} + A_s G_s \sin\eta_{zs}\right)e^{ik_x x}e^{-i\omega t} \tag{26}$$

A similar expression is obtained at $z = -L$. Since we are using a fluid result for the plasma dielectric, the present calculation corresponds to the limit $\omega \gg v_{the}/L$ in which the two sheaths are uncorrelated (electrons cannot communicate between sheaths). Thus, we are studying the effect of the sheath capacitance on a single sheath in this limit.

###   D.   Child-Langmuir Law

In previous sections, the sheath width $\Delta$ was considered an arbitrary input parameter. However, as discussed in paper I and in Ref. 11, the sheath width must be determined self-consistently by requiring that it satisfy the Child-Langmuir (CL) Law[32,33] in the form

$$\Delta = \lambda_D\left(\frac{eV_0}{T_e}\right)^{3/4} \approx \lambda_D\left|\frac{eV_{sh}}{T_e}\right|^{3/4}, \tag{27}$$

where $\lambda_D = (T_e/4\pi ne^2)^{1/2}$ is the Debye length, $V_0 = 3T_e/e + C_{sh}V_{sh}$ is the "rectified" (dc) sheath potential including the Bohm contribution $eV_{Bohm} \approx 3T_e$, and $C_{sh}$ is an order unity rectification coefficient[7] We denote the value of $V_{sh}$ that satisfies the CL Law by $V_{CL}$.

In general, $V_{sh}$ is a complicated nonlinear function of $\Delta$. In addition to the explicit linear dependence on $\Delta$, Eq. (26) will have other $\Delta$-dependent terms in the expressions for $A_f$ and $A_s$. Examples of how to solve for the $\Delta$ and $V_{CL}$ scalings were given in paper I and in Refs. 11, 25-27. We will not treat the self-consistent solution of $V_{CL} = V_{sh}$ here because it will be shown presently that the sheath voltages generated by Eq. (26) are typically small, or at most of order $3T_e$, and enhancement by the sheath-plasma resonance is not expected.

## IV. Analytic and numerical solutions

We will now apply the formalism of Sec. III to compute an analytic solution for the rf fields and sheath potential for FW-driven sheaths. The characteristics of the FW are known, so we specify the FW amplitude $A_f \equiv \hat{E}_y$ and solve for the SW amplitude $A_s$. One can obtain an analytic solution by expanding about the tenuous plasma limit ($\varepsilon_\perp \to 1$, $\varepsilon_x \to 0$ and $\eta_{zf} \to \pi/2$). The following definitions are useful:

$$\delta \equiv \pi/2 - \eta_{zf}, \qquad \eta_0 \equiv \omega L/c \quad . \tag{28}$$

Expanding in $\delta$, we find that $\tan\eta_{zf} \approx 1/\delta$, $\cos\eta_{zf} \approx \delta$, $\sin\eta_{zf} \approx 1$ and $n_{zf} = \eta_{zf}/\eta_0 \approx (\pi/2\eta_0)$ to lowest order in $\delta$. We employ this expression for $n_{zf}$ in the lowest-order dispersion relations for the FW and SW in the tenuous plasma limit, viz.

$$n_f^2 = 1 \implies n_x^2 = 1 - (\pi/2\eta_0)^2 \quad , \tag{29}$$

$$n_{zs}^2 = 1 \implies \eta_{zs} \equiv n_{zs}\,\eta_0 \approx \eta_0 \quad . \tag{30}$$

Using these results in the definition of $Q_j$ [see Eq. (22)], it can be shown that $Q_f \approx \varepsilon_x / n_x^2$ and $Q_s \approx -n_x^2 / \varepsilon_x$, so that the small parameter in this expansion is $\varepsilon_x / n_x^2 \ll 1$. We treat the sheath parameter $p \equiv n_x^2(\Delta/L)$ of order unity. Expanding Eq. (21), one obtains

$$\delta \approx \frac{Q_f}{Q_s} \frac{(\pi/2)\,p}{\left(1 + p\,\eta_0 \tan\eta_0\right)} = -\frac{\pi}{2} \frac{\varepsilon_x^2}{n_x^4} \frac{p}{\left(1 + p\,\eta_0 \tan\eta_0\right)} \quad . \tag{31}$$

The SW amplitude is given by Eq. (20), which completes the solution for the rf fields:

$$A_s = -\frac{A_f \cos\eta_{zf}}{\cos\eta_{zs}} \approx -\hat{E}_y \frac{\delta}{\cos\eta_0}$$

$$= \hat{E}_y \frac{\pi}{2} \frac{\varepsilon_x^2}{n_x^4} \frac{p}{\left(\cos\eta_0 + p\,\eta_0 \sin\eta_0\right)} \tag{32}$$

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Using these results in Eq. (24) for $\varepsilon_{\parallel}E_{\parallel}$, we find that both the FW and SW contributions are of the same order in the small parameter $\varepsilon_{\times}/n_x^2$. To lowest order in the expansion, the coefficients are given by

$$G_f \approx \frac{-\pi}{2\eta_0 n_x} \tag{33}$$

$$G_s \approx \frac{n_x n_{zs}(1-n_{zf}^2)}{\varepsilon_{\times}^2} = \frac{n_x^3}{\varepsilon_{\times}^2} \tag{34}$$

Since the amplitudes $A_j$ and the coefficients $G_j$ are known, we can evaluate the voltage drop across the sheath in Eq. (26). The result is

$$
\begin{aligned}
V_{sh}(L) &= \varepsilon_{\times}\Delta\hat{E}_y \frac{n_{zf}}{n_x}\left(1 - \frac{p\eta_0 \tan\eta_0}{1+p\ \eta_0 \tan\eta_0}\right) \\
&= \frac{\varepsilon_{\times}\Delta\hat{E}_y}{1+p\ \eta_0 \tan\eta_0}\ \frac{n_{zf}}{n_x}
\end{aligned} \tag{35}
$$

where $p \equiv n_x^2(\Delta/L)$, $n_{zf} = (\pi/2\eta_0)$, $n_x = \sqrt{1-(\pi/2\eta_0)^2}$ and $\eta_0 \equiv \omega L/c$. Note that a cancellation occurs between the FW and the SW terms in the first line of Eq. (35). In the limit $p \rightarrow \infty$, the sheath potential vanishes. If $p\ \eta_0\ \tan\eta_0 = -1$, one obtains the sheath-plasma resonance discussed in Appendix A; however, this is unlikely to be important in practice because $\eta_0 \sim 1$ and $p \ll 1$ are typical values for realistic situations. If $p\ \eta_0\ \tan\eta_0 \ll 1$, the SW contribution is small, and the sheath potential in this case is driven by the FW alone. Taking this limit, and assuming that $n_{zf}/n_x \sim 1$, we obtain the following order of magnitude estimate for the sheath potential

$$V_{sh}(L) \approx \varepsilon_{\times}\Delta\hat{E}_y\ . \tag{36}$$

When $eV_{sh} \gg 3T_e$, the sheath width $\Delta$ is given by the Child-Langmuir Law, as discussed in Sec. III D. In what follows, we will use the lower bound estimate, $\Delta = \lambda_D$, because the rf contribution to the sheath potential turns out to be small.

We can now estimate the typical sheath potential due to the $\varepsilon_x$ effect. The full set of equations [Eqs. (8) and (21)] without the $\delta$ expansion was solved numerically and compared with the estimate provided by this analytic solution. The following base case was used: deuterium plasma ($Z = 1$, $\mu \equiv m_i / m_p = 2$) with B = 3 T, $T_e = 20$ eV, L = 50 cm, $\omega = 2\Omega_i$, and $\hat{E}_y = 250$ V/cm (FW field ). (For these parameters, we note that the LH resonance occurs at $n_e = 6.8 \times 10^{10}$ cm$^{-3}$, the FW cut-off occurs at $n_e = 7.1 \times 10^{11}$ cm$^{-3}$, and the sheath plasma resonance does not occur.) For $n_e < 10^{11}$ cm$^{-3}$, the estimate in Eq. (36) is good to within 10% but the sheath voltages are low, $V_{sh} < 10$ V. Above the FW cut-off, the tenuous plasma result in Eq. (36) gives an over-estimate of the sheath voltage. From the numerical solution one finds that $V_{sh}$ increases with density, reaching 40 V at $n_e = 10^{13}$ cm$^{-3}$, but this contribution is much smaller than the rf sheath voltage due to magnetic field tilt discussed in paper I. The numerical work confirms that the SW contribution to Eq. (35) is small over the density range considered ($10^{10}$ cm$^{-3}$ < $n_e$ < $10^{13}$ cm$^{-3}$). Thus, with the FW providing the dominant $E_{\parallel}$ driving the sheaths, it is not surprising that the resulting sheath voltages are small.

In summary, in paper I and the present paper, we have studied rf sheaths computed including the effect of the sheath BC on the FW launched by an ICRF antenna. Here, we have considered additional finite density effects not contained in paper I. (Previously, the electron density contribution in $\varepsilon_{\parallel}$ was retained, but the ion contributions to $\varepsilon_{\perp}$ and $\varepsilon_x$ were neglected.) Here, we find that the plasma anisotropy effect in $\varepsilon_x$ provides an additional drive for the rf sheath voltage, but it is small compared to the effect considered in paper I, viz. the magnetic field tilt $b_y$.

## V. RF sheath power dissipation

This paper concerns rf sheath behavior at high density. One effect is the FW mechanism proportional to $\varepsilon_x$ for driving rf sheaths potentials, discussed in Secs. III and

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IV. Another important effect is the dissipation of power (by several mechanisms) in the antenna and other surrounding structures[34,35] by rf sheaths, which is generally important only at significant density. In this section, we apply our antenna model to estimate the magnitude of power dissipation in antenna sheaths and to determine the most important loss channel. We consider the case of monopole phasing, assume the dominant sheath-drive mechanism (discussed in paper I) and take the limit $(\Delta/L)n_{xf}^2 \gg 1$, in which the vacuum and plasma sheath models agreed in paper I. In this limit, $E_\parallel$ is given by $E_\parallel = V_{sh}/(\Delta\varepsilon_\parallel)$ in the plasma and by $E_\parallel = V_{sh}/\Delta$ in the sheath.. (The result for $E_\parallel$ in the plasma is exact in the tenuous plasma limit and approximate in the high density limit.) Unless otherwise noted, we use the following base case for doing numerical estimates: a deuterium plasma ($Z = 1$, $\mu \equiv m_i/m_p = 2$) with B = 3 T, $n_i = n_e = 10^{11}$ cm$^{-3}$, $T_e = 30$ eV, L = 50 cm, $\omega = 2\Omega_i$, and $V_{rf} = 600$ V.

The first sheath power dissipation mechanism is the *acceleration of ions* in the sheaths and the subsequent dissipation of that energy in the material boundary.[35] The ion power dissipation ($P_i$) per unit area is given by $S_i \equiv P_i/A_\perp = n_i c_s (eV_0) \approx C_{sh} n_i c_s (eV_{sh})$ for $eV_{sh}/T_e \gg 1$, where $C_{sh}$ is an order unity rectification coefficient.[7] Using $n_i = n_e$ and introducing the distance 2L between sheaths, we can write this expression as $S_i = C_{sh}(eV_{sh})n_e \nu_\parallel (2L)$ with $\nu_\parallel = c_s/(2L)$. For the base case parameters, we obtain

$$S_i\left(\frac{kW}{cm^2}\right) = 0.04\frac{n_i}{10^{11}\,cm^{-3}} \qquad . \qquad (37)$$

Thus, the sheath power dissipation due to this process is very small for the low densities near the Faraday screen, but a significant heat flux can occur at higher densities, $n_i \approx 10^{12}$ cm$^{-3}$.

We now turn to the electron dissipation mechanisms. The power per unit area for the nonlinear electron heating mechanisms can be put in the general form

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$$S_e = \tfrac{1}{2} m_e (\delta v)^2 n_e \, \nu \, 2L \quad , \tag{38}$$

where $\delta v$ and $\nu$ are the characteristic velocity kick and power dissipation rate defined below, and $2L$ is the distance between sheaths.

The first example is *collisional heating of electrons* in the plasma between the sheaths, due to collisions with neutrals.[36,37] (Neutral collisions can also occur in the sheath itself when the electron mean free path is short enough, $S_e = \tfrac{1}{2} m_e (\delta v)^2 n_e \, \nu \, 2L$ $\lambda_e < \Delta$, but this regime is not expected to apply for fusion plasmas.) The typical velocity in the bulk plasma is the jitter velocity of the electrons in the rf field, $\delta v \to v_{rf} = e E_\parallel /(m_e \omega)$, where $E_\parallel$ is the parallel component of the rf electric field in the plasma region. Our antenna model gives the typical field $E_\parallel = V_{sh}/(\Delta \varepsilon_\parallel)$. The dissipation rate is given by the electron-neutral collision frequency, $\nu \to \nu_{en} = n_0 \sigma_{en} v_e$, where $v_e$ is the electron thermal speed and $\sigma_{en} = 5 \times 10^{-15}$ cm$^2$, approximately independent of species.[38] An estimate for the neutral density is given by the recycling condition $n_0 v_0 = n_i v_i$, where $v_0$ and $v_i$ are the thermal velocities of neutrals and ions, and a recycling coefficient of unity has been assumed. Other mechanisms, like outgassing and external gas sources, can give even higher neutral densities near the antenna. For the base case parameters, we obtain the following scaling of $S_c$, the power per unit area dissipated by electron-neutral collisions, with the plasma and neutral densities

$$S_c \left( \frac{kW}{cm^2} \right) = 0.003 \frac{n_e}{10^{11}\,cm^{-3}} \frac{n_0}{10^{14}\,cm^{-3}} \quad . \tag{39}$$

Another mechanism for electron heating is *Fermi acceleration*,[36,37,39,40] where the electrons are heated stochastically by bouncing off the oscillating sheaths at both ends of the field lines. An order of magnitude estimate of stochastic heating can be obtained by using the rate $\nu \to v_e/(2L)$ and the velocity kick $\delta v \to \omega \Delta$, where the sheath width $\Delta$ is given by the Child-Langmuir Law. Numerical calculations show that the estimate $\delta v \approx \omega \Delta$ is valid for $\omega \Delta < v_e$.[40] Using these estimates, the power dissipation per unit

area for Fermi acceleration (stochastic electron heating) is $S_F = \frac{1}{2} m_e (\omega \Delta)^2 n_e v_e$, which is independent of density. For the base case parameters, we obtain

$$S_F \left( \frac{kW}{cm^2} \right) = 0.001 \quad . \tag{40}$$

Thus, the stochastic electron power dissipation exceeds that due to collisional heating at sufficiently low neutral density (gas pressure).[41] For $n_e = 10^{11}$ cm$^{-3}$, the condition $S_F > S_c$ requires $n_0 < 3 \times 10^{13}$ cm$^{-3}$. For these parameters, the ion power dissipation is the largest contributor to the power losses. The design limit[42] for the heat load on material surfaces in ITER (assuming normal incidence) is about 1-2 kW / cm$^2$. The estimates in Eqs. (37) – (40) show that sheath power dissipation could be a problem on parts of the antenna structure that encounter large plasma densities ($\sim 10^{12}$ cm$^{-3}$).

Finally, we point out that neutrals can influence the sheath formation if the ion mean free path for neutral collisions, $\lambda_{in} = v_i / v_{in} = 1/(n_0 \sigma_{in})$, is comparable to the sheath width $\Delta$. This effect has been studied in the plasma processing literature (e.g. see Ref. 43) These effects have been neglected in the present work because $\lambda_{in} > 2$ cm for $n_0 < 10^{14}$ cm$^{-3}$ and the condition $\lambda_{in} > \Delta$ is easily satisfied.

## VI. Summary and Conclusions

This paper investigated a new mechanism for rf sheath formation on antennas. Combining the work in Ref. 9 (paper I) and the present work (paper II), we have formulated two analytic models of the coupling of a FW and SW by the rf sheath BC, including the effects of magnetic field line tilt ($V_{sh} \propto b_y \hat{E}_y$) and of the full plasma dielectric tensor ($V_{sh} \propto \varepsilon_\times \Delta \hat{E}_y$). For typical parameters, the magnetic field misalignment is the stronger effect. A calculation retaining both effects simultaneously was also carried out, but the additional effects do not add additional insight in the assumed ordering and were not discussed here. The present paper shows that the finite-$\varepsilon_\times$ sheath drive is due to

the FW rather than the SW; therefore, it is weak and the earlier calculation in paper I is not missing any essential density-related effects. Both calculations may be useful for benchmarking the next generation of antenna and SOL codes that incorporate sheath effects self-consistently through the sheath BC.

The question of how these models relate to the vacuum-field sheath approximation commonly used in present rf codes was discussed in Sec. I. Present ICRF antenna codes evaluate the SW driven by the magnetic field tilt (and other effects outside the scope of the analytic models) but do not include the effects treated in papers I and II. The SW considered in our models owes its existence to the sheath BC; it has the same maximum value and potentially greater radial extent than the SW considered in the codes, but in general its amplitude is smaller. It was shown in paper I that the two slow waves yield the same sheath potential in the limit $(\Delta / L)\, n_x^2 \to \infty$.

From papers I and II we conclude that the sheath boundary condition plays an essential role in describing the interactions of antenna near fields with the antenna structure, and that minimally the tenuous-plasma dielectric should be retained for the plasma model in the vicinity of the antenna. Retaining both mechanisms (i) and (ii) discussed in the introduction, in addition to the effect of antenna corners, protrusions and the Faraday screen, are all required for quantitative results and will clearly require advanced numerical codes and additional work.

We conclude by pointing out that for general geometry the two terms in the sheath BC stand in the relation

$$1 \;:\; \mathbf{b} \cdot \mathbf{s} \, (k_{\parallel} L_{\parallel}) \left( \frac{\Delta}{L_{\parallel}} n_t^2 \right) \quad , \tag{41}$$

where the first term represents the metal wall BC and the second term gives the sheath capacitance contribution. Here, $\mathbf{b} = \mathbf{B}/B$, $\mathbf{s}$ is the normal to the sheath, $\Delta$ is the sheath

width, and $\mathbf{n}_t$ is the tangential part of the index of refraction $\mathbf{n} = \mathbf{k}c/\omega$. In the present work, $\mathbf{n}_t = n_x \hat{e}_x$ because we assumed $n_y = 0$ and $k_\parallel L_\parallel$ is order unity for the lowest mode. Thus, $(\Delta/L)\, n_x^2$ emerges naturally as a parameter representing the effect of the sheath capacitance in the BC. (As an aside, we note that $n_x^2 \sim \varepsilon_\parallel$ in the conventional SW ordering considered e.g. in Refs.11, 26, and 27, so in this case the natural parameter is equivalent to $\Lambda \equiv -\varepsilon_\parallel \Delta/L$, the sheath capacitance parameter defined in paper I.

Other topics related to the theme of calculating antenna sheath effects were also discussed in this series of papers. In paper I, the effect of phasing dependence of the current straps on the sheath voltage was examined. Also, it was shown how to compute the sheath width self-consistently using the Child-Langmuir Law. In Sec. V of paper II, we surveyed mechanisms by which the sheaths can dissipate rf power, and discussed the role of neutrals.

Finally, in Appendix A of this paper, a short overview is given of a number of related sheath calculations using the sheath BC, which help to put the present work in context. It is shown in Appendix A that all of these sheath calculations have a number of features in common, and these features have important consequences for understanding the interaction of ICRF antennas with the edge plasma.

## Acknowledgements

## Appendix A  Overview of sheath models

In this Appendix, we put the present work into context by giving an overview of a series of related calculations illustrating rf sheath physics by means of the sheath BC (SBC).[9,11,25-27] These calculations differ in the types of waves driving the sheath formation and in the assumed magnetic geometry. There are two physical situations which must be distinguished: (1) the FW propagates (or evanesces) across a magnetic field and a parasitic SW is generated by the mismatch of the equilibrium magnetic field orientation with the bounding surface;[9,10,26] or (2) the SW propagates (or evanesces) across magnetic field lines and directly induces sheath potentials at the material boundaries.[11,26,27] In all of these cases, another important distinction is whether the sheath-plasma resonance[36,44-46] plays a role in enhancing the sheath potential. This depends on the range of wavenumbers involved in the wave propagation, and on the wave polarization.

First, we discuss the SW case. In Ref. 11, we studied the propagation of an electrostatic SW (electron plasma wave) perpendicular to **B** and along the axis of a waveguide using the SBC at the waveguide walls. This analytic calculation showed how the sheaths impose a localization condition (along **B**) on the mode, and that plasma screening of the rf field occurs when $\varepsilon_{\parallel} \gg 1$. Other sheath effects (e.g. the sheath plasma wave) become important when the sheath capacitance parameter $\Lambda \equiv -\varepsilon_{\parallel}(\Delta/L)$ is order unity ($\Delta$ is the sheath width and 2L the distance between the waveguide walls). The effect of sheath power dissipation due to ion acceleration in the sheaths was also included in the formalism of Ref. 11. It was shown that the plasma-sheath system conserves energy in the calculation of the self-consistent Child-Langmuir (CL) potential. Power lost to the sheaths reduces the wave amplitude and Poynting flux of the electron plasma wave such that energy is conserved.

In Refs. 26 and 27, we studied the behavior of a SW launched by the antenna. Below the LH density, the SW propagates in the form of a resonance cone[26] (RC) The RC propagates out to the sheaths and must satisfy the SBC there. It was shown in Ref. 26 that the self-consistent (Child-Langmuir) sheath potential generated at the wall is a significant fraction of the RC voltage at the antenna when $\Lambda_{RC} \equiv \varepsilon_{||}(\Delta/a_{||}) >> 1$, where $a_{||}$ is the dimension of the RC structure along the magnetic field at the antenna. Since the RC propagates essentially parallel to **B**, this calculation provides a mechanism for conveying antenna voltage to distant surfaces around the tokamak.[22] Above the LH density, the SW does not propagate, but for sufficiently close limiters the SBC introduces a new mode, the sheath-plasma wave, which can propagate away from the antenna and carry the antenna voltage to surrounding surfaces.[27] A self-consistent calculation of the rf-sheath width yields the resulting sheath voltage in terms of the amplitude of the launched slow wave, plasma parameters and connection length.

Sheath formation in the FW cases involves secondary (parasitic) coupling of the FW to the SW, because the polarization of the propagating FW has $E_{||}=0$. In Refs. 10 and 25 we considered the "far field sheath" problem in which rf wave energy encounters distant (compared to the FW wavelength) surfaces. The fast waves are assumed to be incident on a conducting boundary not aligned with a flux surface (i.e. $\mathbf{s} \cdot \mathbf{b} \neq 0$, where $\mathbf{s}$ is the normal to the conducting surface). In this case, the magnetic field orientation is such that the FW cannot satisfy the sheath BC without coupling to the SW,[10,47] and rf sheaths are generated by the resulting SW. Recently, an analytic approach[25] to this problem was formulated using a wave scattering formalism with the SBC determining the coupling coefficients of the reflected waves (FW + SW). As in the SW problems described previously, the self-consistent CL sheath potential was calculated and was shown to have a large effect on the solutions. This nonlinear constraint introduced multiple roots, some of which had large sheath potentials (because of sheath-plasma

resonance) even when the FW amplitude at the wall was modest. This problem is relevant to fast waves which propagate through the plasma under conditions of low single pass absorption or waves with FW polarization that propagate around the SOL and encounter material boundaries.

In Ref. 9 (paper I) and in the present work (paper II), we consider the "near field sheath" problem for a FW ICRF antenna. (In the assumed waveguide geometry, this calculation generalizes Ref. 11 to electromagnetic waves and to general parity.) As in all of these calculations, magnetic geometry plays a crucial role. In paper I it was shown that the FW is coupled to the SW by the sheath BC when the magnetic field orientation is not properly aligned with the antenna. In the present paper, we show that high density plasmas have a second coupling mechanism: the off-diagonal plasma dielectric tensor element $\varepsilon_\times$ couples the FW and SW and causes a (typically small) enhancement of the sheath, which persists even when the magnetic field is perfectly aligned. These analytic calculations exhibit important *qualitative* features inferred by past antenna sheath studies, such as dependence of the sheath potential on the antenna phasing (which determines the parity of the rf fields and the degree of cancellation of the magnetic flux driving the sheath voltage[8]). They provide for the first time a way to incorporate the important effect of sheath capacitance on the plasma-sheath system. Future work, incorporating this physics into models with more realistic geometry, will be required to obtain quantitative evaluations of the this effect for real antennas.

To summarize, all of these calculations taken together make the following points:

(1) Sheaths are generated by the magnetic field mismatch with material boundaries and by the anisotropy of the plasma dielectric tensor.

(2) When finite plasma density near the sheath is taken into account, the sheath capacitance causes a voltage redistribution along the B field.

(3) The condition for strong sheath effects in the FW-driven case (papers I and II) is $n_x^2(\Delta/L) > 1$, or for the conventional SW case, $\Lambda \equiv -\varepsilon_{\parallel}(\Delta/L) > 1$, where L is some characteristic distance along the field lines (problem dependent).

(4) The sheath boundary condition introduces a new class of modes, called sheath-plasma waves, and a new resonance; the resonance can enhance the sheath potential in some cases.

## Appendix B  Dispersion relations and SW eigenmode

In this Appendix, we review some details of the dispersion relations used in the main text. The general dispersion relation for the coupled fast and slow waves is given by

$$\det \mathbf{L}_0 = \begin{vmatrix} \varepsilon_{\perp} - n_z^2 & -i\varepsilon_{\times} & n_x n_z \\ i\varepsilon_{\times} & \varepsilon_{\perp} - n^2 & 0 \\ n_x n_z & 0 & \varepsilon_{\parallel} - n_x^2 \end{vmatrix} = 0 \quad, \tag{B1}$$

$$\Rightarrow \quad (\varepsilon_{\parallel} - n_x^2)[(\varepsilon_{\perp} - n_z^2)(\varepsilon_{\perp} - n^2) - \varepsilon_{\times}^2] = n_x^2 n_z^2 (\varepsilon_{\perp} - n^2) \quad, \tag{B2}$$

where $n^2 = n_x^2 + n_z^2$ and $n_z \equiv n_{\parallel}$, $n_x \equiv n_{\perp}$. For specified $n_x$, Eq. (B2) is a second order equation for $n_z^2$ and there are two roots which lie on the FW and SW branches.

The two waves uncouple for large $\varepsilon_{\parallel}$. The FW ordering is $n_x^2 \sim n_z^2 \sim \varepsilon_{\perp} << \varepsilon_{\parallel}$. Taking the limit $\varepsilon_{\parallel} \to \infty$ in Eq. (B2), the FW dispersion relation becomes

$$(\varepsilon_{\perp} - n^2)(\varepsilon_{\perp} - n_z^2) - \varepsilon_{\times}^2 = 0 \quad . \tag{B3}$$

In the "tenuous plasma" limit considered in paper I (where $\varepsilon_x = 0$, $\varepsilon_{\perp} = 1$, $|\varepsilon_{\parallel}| >> 1$), the FW dispersion relation in Eq. (B3) reduces to

$$n_f^2 = \varepsilon_{\perp} \quad . \tag{B4}$$

Returning to the arbitrary density case, the usual SW ordering is $n_z^2 \sim \varepsilon_{\perp} << n_x^2 \sim \varepsilon_{\parallel}$. Taking the limit $n_x^2 \sim \varepsilon_{\parallel} \to \infty$ in Eq. (B2), we obtain the SW dispersion relation,

$$\varepsilon_\parallel\left(n_z^2 - \varepsilon_\perp\right) + n_x^2 \varepsilon_\perp = 0. \tag{B5}$$

It is important to point out that the usual SW ordering is not relevant to the calculation described in Sec. III. In order for the FW to satisfy the sheath BC at each point in x, it must couple to a wave with SW polarization and this wave must have $n_x = n_{xf}$. Thus, in the present problem the SW satisfies the same ordering as the FW, i.e. $n_x^2 \sim n_z^2 \sim \varepsilon_\perp \ll \varepsilon_\parallel$ and also satisfies the dispersion relation in Eq. (B3). As noted above, for a given $n_x$ this is a second order equation for $n_z^2$ and has both FW and SW roots. Writing this equation in the form $(X - X_f)(X - X_s) = 0$ with $X_f \equiv n_{zf}^2$, $X_s \equiv n_{zs}^2$, expanding as $X^2 - X(X_f + X_s) + X_f X_s = 0$, and comparing coefficients with Eq. (B3) one obtains two relationships between the FW and SW roots

$$n_{zs}^2 = 2\varepsilon_\perp - n_x^2 - n_{zf}^2 \tag{B6}$$

and

$$n_{zf}^2 n_{zs}^2 = \varepsilon_\perp\left(\varepsilon_\perp - n_x^2\right) - \varepsilon_\times^2 \quad . \tag{B7}$$

In the main text, the dispersion relations for the FW and SW are solved as follows. The wavenumber $n_{zf}$ is determined by the sheath BC, and Eq. (B3) is solved for $n_{xf}^2$ with $n_z^2 = n_{zf}^2$. Finally, Eq. (B3) with $n_{xs} = n_{xf}$ (required by the sheath BC) is solved for $n_{zs}^2$ using the constraint (B6) or (B7). Using Eq. (B3) to solve for $\varepsilon_\perp - n_{zf}^2$ in Eq. (B6), one obtains the result that

$$n_{zs}^2 - \varepsilon_\perp = \varepsilon_\perp - n_f^2 = \frac{\varepsilon_\times^2}{\varepsilon_\perp - n_{zf}^2} \quad . \tag{B8}$$

This is a very useful form of the dispersion relation. If $\varepsilon_\times = 0$, it shows that the FW and SW dispersion relations reduce to $\varepsilon_\perp - n_f^2 = 0$ and $n_{zs}^2 - \varepsilon_\perp = 0$, respectively, as in paper I. For the general case, Eq. (B8) shows that specifying $n_{zf}$ and the plasma dielectric is sufficient to determine $n_{zs}$. Finally, we note that the rhs of Eq. (B8) tends to be small,

even at high density where $\varepsilon_\times \sim 1$, because $n_{zf}^2 \gg 1$ for typical parameters. Thus, the character of the SW solution is approximately determined by whether the density lies below or above the LH resonance ($\varepsilon_\perp = 0$).

We have seen that the point where $n_{zs}^2 = 0$ is approximately given by $\varepsilon_\perp = 0$, although there are small corrections from the $\varepsilon_\times$ term in Eq. (B3) that are retained in the numerical solution. For densities low enough that $\varepsilon_\perp > 0$, $n_{zs}$ is real and the SW eigenmode has a sinusoidal z dependence. It is a global mode, spanning the antenna region along z. For higher density, $\varepsilon_\perp < 0$, $n_z$ is imaginary, the SW eigenmode has a hyperbolic z dependence, and the SW fields are evanescent in z away from the boundaries at $z = \pm L$. As the density increases and $n_z \to i\,\infty$, the SW fields become concentrated near the sheaths.

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