# Parallel Kelvin-Helmholtz mode benchmark

J. R. Myra, D. A. Baver

*Lodestar Research Corp., 2400 Central Ave. P-5, Boulder, Colorado 80301*

M. V. Umansky

*Lawrence Livermore National Laboratory, Livermore, CA 94550*

## *A. Introduction*

This test was devised to verify the ability of the 2DX eigenvalue code to correctly solve a fluid model relevant to edge turbulence in tokamaks, viz. the parallel Kelvin Helmholtz mode.[1,2] Since the functionality of the 2DX code depends on both the source code itself and the input file defining the system of equations to solve (structure file), this test demonstrates both. Similar tests have been performed using other physics models. Moreover, since the structure file for these tests represents a subset of a more general 6-field model, many of the terms in that test are also verified. A more detailed description of the 2DX code can be found in Ref. 3.

The present test compares 2DX results to exact semi-analytic results based on a local limit of the eigenvalue problem. The semi-analytic results are equivalent to the numerical solution of a polynomial dispersion relation, i.e. they are obtained without discretization.

## *B. Parallel Kelvin-Helmholtz 4-field model*

A 4-field model containing the physics of the parallel Kelvin-Helmholtz (pKH) mode is[1,2]

$$\gamma\nabla_\perp^2\delta\Phi + \gamma\frac{T_i}{n}\nabla_\perp^2\delta n = -u\nabla_{||}\nabla_\perp^2\delta\Phi + \frac{B^2}{n}\partial_{||}\delta J + \mu_{ii}\nabla_\perp^4\delta\Phi \tag{1}$$

$$\gamma\delta n = -u\nabla_{||}\delta n - \delta v_E\cdot\nabla n - n\partial_{||}\delta u + \partial_{||}\delta J \tag{2}$$

$$\gamma\delta u = -u\nabla_{||}\delta u - \delta\mathbf{v}_E\cdot\nabla u - \frac{T_e+T_i}{n}\nabla_{||}\delta n + \partial_{||}\mu_{||}\nabla_{||}\delta u \tag{3}$$

$$\gamma\delta J = -u\nabla_{||}\delta J - \nu_e\,\delta J - \mu n\nabla_{||}\delta\Phi + \mu T_e\nabla_{||}\delta n \tag{4}$$

where we work in Bohm-normalized variables with times normalized to $1/\Omega_i$, lengths normalized to $\rho_s$, temperature and electrostatic potential $\Phi/e$ to a reference value of $T_e$, and density to a reference value of $n_e$. Here $\mu = m_i/m_e$, and for any Q, $\partial_{||}Q = B\nabla_{||}(B^{-1}Q)$, $\partial_r = RB_p\partial/\partial\psi$, $k_b$ is the binormal component of the perpendicular wavenumber; other symbols have their usual meanings. In particular $\delta\Phi$, $\delta n$, $\delta u$ and $\delta J$

are respectively the perturbations of electrostatic potential, density, parallel velocity and parallel current. See Ref. 3 for a complete description of the six-field model.

In the local limit, and choosing local values for $\Omega_i$, $\rho_s$, $T_e$ and $n_e$, we set n = B = $T_e$ = 1, $T_i = \tau$, and working in the frame u = 0, these equations reduce to

$$\gamma\delta\Phi + \tau\gamma\delta n = -i\frac{k_{||}}{k_\perp^2}\delta J - \mu_{ii}k_\perp^2\delta\Phi \tag{5}$$

$$\gamma\delta n = ik_b(\partial_r n)\delta\Phi - ik_{||}\delta u + ik_{||}\delta J \tag{6}$$

$$\gamma\delta u = ik_b(\partial_r u)\delta\Phi - (1+\tau)ik_{||}\delta n - \mu_{||}k_{||}^2\delta u \tag{7}$$

$$\gamma\delta J = -\nu_e\,\delta J - \mu i k_{||}\delta\Phi + \mu i k_{||}\delta n \tag{8}$$

We can eliminate the $\gamma\delta n$ term from vorticity to obtain

$$\gamma\delta\Phi = -i\tau k_b(\partial_r n)\delta\Phi + i\tau k_{||}\delta u - i\tau k_{||}\delta J - i\frac{k_{||}}{k_\perp^2}\delta J - \mu_{ii}k_\perp^2\delta\Phi \tag{9}$$

In Bohm dimensionless units let $\omega_{*e} = k_b(\partial_r n) = k_b/L_n$, $\omega_s = k_{||}$, $\omega_{*u}$ = $k_b/L_u$, $\omega_{*i} = \tau k_b(\partial_r n) = \tau k_b/L_n$. This yields the following set of local equations

$$\gamma\delta\Phi = (-i\omega_{*i} - \mu_{ii}k_\perp^2)\delta\Phi + i\tau\omega_s\delta u + \left(-i\tau\omega_s - i\frac{k_{||}}{k_\perp^2}\right)\delta J \tag{10}$$

$$\gamma\delta n = i\omega_{*e}\delta\Phi - i\omega_s\delta u + i\omega_s\delta J \tag{11}$$

$$\gamma\delta u = i\omega_{*u}\delta\Phi - i(1+\tau)\omega_s\delta n - \mu_{||}k_{||}^2\delta u \tag{12}$$

$$\gamma\delta J = -i\mu k_{||}\delta\Phi + i\mu k_{||}\delta n - \nu_e\,\delta J \tag{13}$$

We note parenthetically, that the parallel KH mode *exists* in a much simpler 2-field model obtained from the above by dropping the vorticity and $\delta J$ equations, assuming Maxwell-Boltzmann electrons, dropping the $\delta J$ term in continuity, and the $\mu_{||}$ term in the $\delta u$ equation. This simpler model yields instability for $L_u < L_n$, but has the difficulty that in the plane where both $k_b$ and $k_{||}$ vary to arbitrarily large values, the growth rate increases without bound. Thus the simpler 2-field is neither suitable for benchmarking tests or physics exploration with a discretized code. (If the 2-field model were used in a discretized code, the simulations would be dominated by grid-scale modes regardless of the resolution employed.) The 4-field model with finite $\mu_{||}$ solves this problem.

### *C. Results in the local limit*

We choose parameters from the C-Mod QC-mode case, except for $L_u$ the gradient scale length of u, which is varied from strongly unstable ($L_u = 5$) to stable ($L_u = \infty$). The base case parameters in dimensionless Bohm units are $L_n = 12.7$, $\tau = T_i/T_e = 1$, $\nu_e = 0.34$, $\mu = 3600$, $\mu_{ii} = 0.001$, $\mu_{||} = 2000$. The model is that of Eqs. (10) – (13). The equations were solved using Mathematica to generate local results for comparison with 2DX numerical results.

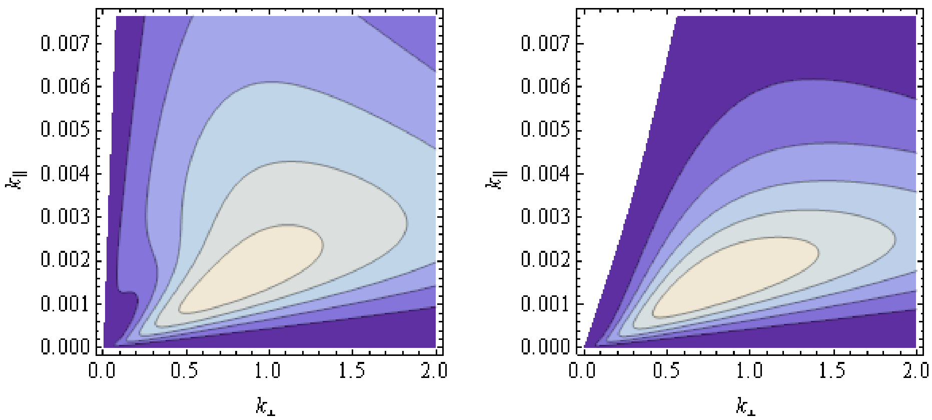


Fig. 1 Growth rate contours for the parallel KH and DW instabilities, for base case parameters (left) and with $1/L_u = 0$ (right). The range of $k_{||}R$ is 40.

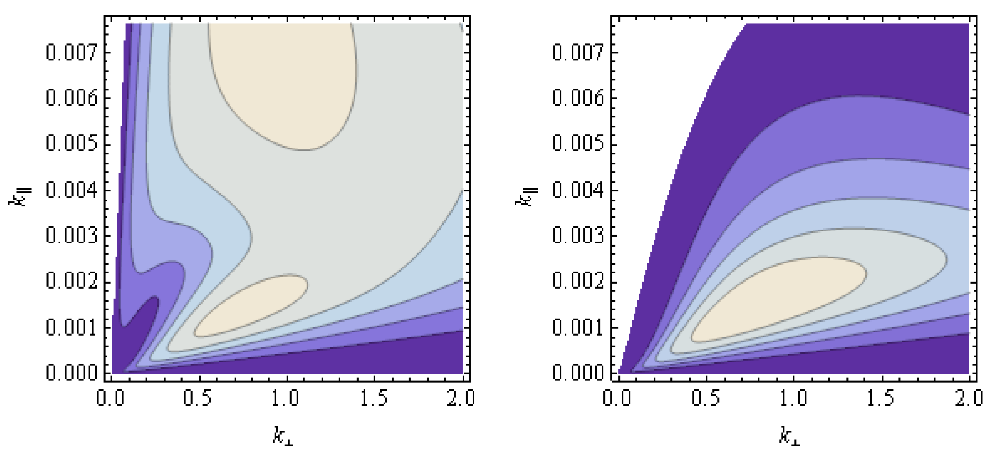


Fig. 2 Same comparison as Fig. 1 except that now $\mu_{||} = 400$. Base case $L_u$ (left) and $1/L_u = 0$ (right).

Results for the base case parameters are shown in Fig. 1. Because of the relatively large (but experimentally realistic) value of $\mu_{||}$, the parallel KH and drift wave (DW) instabilities merge, so the effect of the pKH drive $L_u$ is only to slightly broaden the instability contours.

Fig 2. shows the same comparison for the case $\mu_{||} = 400$. (We cannot take $\mu_{||} = 0$ because then the spectrum never saturates as $k_{||}$ increases.) Now there are two distinct unstable branches: the high $k_{||}$ pKH, and the low $k_{||}$ DW.

## *D. The parallel KH benchmark test*

For a good benchmark test that separates the pKH and DW branches, we choose a case with $\mu_{||}$ 5 times smaller than realistic. The parameter set is: (in dimensionless Bohm units) $L_n = 12.7$, $\tau = T_i/T_e = 1$, $\nu_e = 0.34$, $\mu = 3600$, $\mu_{ii} = 0.001$, $\mu_{||} = 400$. The growth rates in dimensional units for the reference $\Omega_i = 1.982\times10^8$/s are shown in Fig. 3. The conversion from dimensionless binormal wavenumber $k_b$ to toroidal mode number n is given by $n = 1.04\times10^3\ k_b$. For this plot, we choose $k_{||base} = 0.0003$ ($k_{||base}R = 1.5687$). This is the fundamental mode. Then we calculate growth rates $\gamma$ for all the m $k_{||base}$ (m = 1, 2, 3, …) and pick the maximum $\gamma$. This is done for $L_u = 5$ (parallel KH mode) and $L_u = 500$ (remnant DW mode).

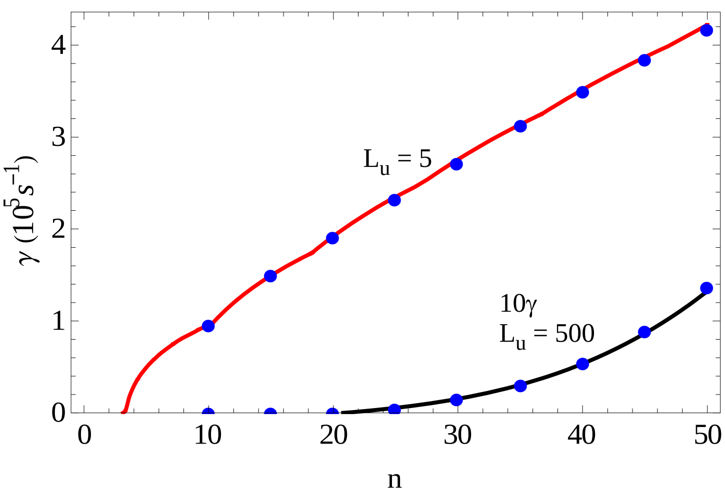


Fig. 3 Semi-analytic growth rates optimized over m $k_{||base}$ for $L_u$ = 5 (red) and 500 (black) (growth rate multiplied by 10). Blue dots are the 2DX results.

| **n** | **γ ($10^5$ $s^{-1}$)**<br>**$L_u$ = 5** | **γ ($10^5$ $s^{-1}$)**<br>**$L_u$ = 500** |
|---|---|---|
| 10 | 0.96151 | -0.01152 |
| 15 | 1.50619 | -0.00577 |
| 20 | 1.93679 | -0.00086 |
| 25 | 2.36788 | 0.00553 |
| 30 | 2.77891 | 0.01543 |
| 35 | 3.16148 | 0.03085 |
| 40 | 3.54409 | 0.05398 |
| 45 | 3.90071 | 0.08727 |
| 50 | 4.25273 | 0.13338 |

Table 1. Table of semi-analytic growth rates for the benchmark case shown in Fig. 3. These are the target results for the benchmark test of the numerical code (2DX).

## *References*


1. P.J. Catto, M.N. Rosenbluth and C.S. Liu, Phys. Fluids **16**, 1719 (1973).
2. F. Schwander et al., J . Nucl. Mater. in press, (2010), doi:10.1016/j.jnucmat.2010.10.073; and refs. therein
3. D. A. Baver, J. R. Myra and M.V. Umansky, Comp. Phys. Comm. **182**, 1610, (2011).