Toroidal ITG benchmark

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A. Introduction

This test was devised to verify the ability of the 2DX eigenvalue code to correctly solve a fluid model relevant to turbulence in tokamaks, viz. the toroidal ion temperature gradient (ITG) mode.[1,2] Since the functionality of the 2DX code depends on both the source code itself and the input file defining the system of equations to solve (structure file), this test demonstrates both. Similar tests have been performed using other physics models. Moreover, since the structure file for these tests represents a subset of a more general 6-field model, many of the terms in that test are also verified. A more detailed description of the 2DX code can be found in Ref. 3.

The present test compares 2DX results to exact semi-analytic results based on a local limit of the eigenvalue problem. The semi-analytic results are equivalent to the numerical solution of a polynomial dispersion relation, i.e. they are obtained without discretization. The 2DX solutions of the ITG problem given here implement the limit of Maxwell-Boltzmann electrons analytically in the model equations. Although 2DX supports more general electron models, this was done to facilitate comparison with simulation codes using the same approximation.

B. The toroidal ITG 3-field model

From the full 6-fld model described in Ref. 3, one obtains a sublimit appropriate to the toroidal ITG mode[1,2] as

$$\gamma\nabla_\perp^2\delta\Phi = -i\omega_{*i}\nabla_\perp^2\delta\Phi + \frac{2B}{n}C_r\delta p + \frac{B^2}{n}\partial_{||}\delta J + \mu_{ii}\nabla_\perp^4\delta\Phi \tag{1}$$

$$\gamma\delta n = -\delta v_E\cdot\nabla n + \frac{2}{B}\left(C_r\delta p_e - nC_r\delta\Phi\right) + \partial_{||}\delta J \tag{2}$$

$$\gamma\delta T_i = -\delta v_E\cdot\nabla T_i + \frac{2T_i}{3n}\partial_{||}\delta J + \frac{4T_i}{3B}\left(\frac{1}{n}C_r\delta p_e - C_r\delta\Phi - \frac{5}{2}C_r\delta T_i\right) \tag{3}$$

$$\gamma\delta J = -\mu n\nabla_{||}\delta\Phi + \mu T_e\nabla_{||}\delta n \tag{4}$$

where we work in Bohm-normalized variables with times normalized to $1/\Omega_i$, lengths normalized to ρ_s, temperature and electrostatic potential Φ/e to a reference value of T_e, and density to a reference value of n_e. Here $\mu = m_i/m_e$, and for any Q, $\partial_{||} Q = B\nabla_{||}(B^{-1}Q)$, $\partial_r = RB_p \partial/\partial\psi$, k_b is the binormal component of the perpendicular wavenumber; other symbols have their usual meanings.[3] In particular $\delta\Phi$, δn, δT_i and δJ are respectively the perturbations of electrostatic potential, density, ion temperature and parallel current.

In the local limit, and choosing local values for Ω_i, ρ_s, T_e and n_e, we set $n = B = T_e = 1$ and $T_i = \tau$. We drop μ_{ii} let $\delta v_E \cdot \nabla n = -ik_b \delta\Phi \partial_r n = i\omega_{*en}\delta\Phi$ where $\omega_{*en} = -k_b \partial_r n$ similarly $\delta v_E \cdot \nabla T_i = i\omega_{*iT}\delta\Phi$ and we note that $\omega_{*iT} = \tau\eta_i\omega_{*en}$. The perturbed pressures in this limit are $\delta p = \delta n(1+\tau) + \delta T_i$ and $\delta p_e = \delta n$. This gives

$$-\gamma k_\perp^2 \delta\Phi = i\omega_{*i} k_\perp^2 \delta\Phi + 2C_r[\delta n(1+\tau) + \delta T_i] + \nabla_{||}\delta J \tag{5}$$

$$\gamma\delta n = -i\omega_{*en}\delta\Phi + 2C_r\left(\delta n - \delta\Phi\right) + \nabla_{||}\delta J \tag{6}$$

$$\gamma\delta T_i = -i\omega_{*iT}\delta\Phi + \frac{2\tau}{3}\nabla_{||}\delta J + \frac{4\tau}{3}C_r\left(\delta n - \delta\Phi - \frac{5}{2}\delta T_i\right) \tag{7}$$

$$\gamma\delta J = \mu\nabla_{||}(\delta n - \delta\Phi) \tag{8}$$

where $C_r = -ik_b/R$ and R is the curvature radius. Finally we take the Maxwell-Boltzmann limit analytically to obtain a 3-field model

$$\gamma\nabla_\perp^2\delta\Phi = -i\omega_{*i}\nabla_\perp^2\delta\Phi + 2C_r[\delta\Phi(1+\tau) + \delta T_i] + \nabla_{||}\delta J \tag{9}$$

$$\gamma\delta\Phi = -i\omega_{*en}\delta\Phi + \nabla_{||}\delta J \tag{10}$$

$$\gamma\delta T_i = -i\omega_{*iT}\delta\Phi + \frac{2\tau}{3}\nabla_{||}\delta J - \frac{10\tau}{3}C_r\delta T_i \tag{11}$$

For the semi-analytical solutions in a slab we also take $\nabla_\perp^2 = -k_\perp^2$, and $\nabla_{||} = ik_{||}$, while these operators retain their full differential form in the 2DX numerical solution.

If we chose $L_n = 1$, i.e. set $\omega_{*en} = k_b$ then frequencies are effectively renormalized to L_n/c_s as in Sandberg[2]. Furthermore we then identify $C_r = -ik_b\varepsilon_n/2$, $\omega_{*iT} = \tau\eta_i k_b$, $\omega_{*i} = -\tau(1+\eta_i)k_b$ [Caution: note the different sign convention in the definitions of ω_{*iT} and ω_{*i}.] Equations (9) – (11) are equivalent to the Sandberg 2-field model,[2] which is obtained by eliminating $\nabla_{||}\delta J$ analytically. We do not perform that elimination here; instead, we retain the model in the 3-field form.

It turns out that the present toroidal ITG problem tests a generalized eigenvalue capability of the 2DX code. Specifically, of the three unknowns, $\delta\Phi$, δT_i and δJ in Eqs. (9) – (11), only $\delta\Phi$ and δT_i appear on the left-hand-side. This results in a generalized eigenvalue problem where the matrix on the left-hand-side is non-invertible. The 2DX code (and the underlying SLEPc[4] eigenvalue solver) have no difficulty with this structure. This observation significantly broadens the class of models that can be treated by 2DX.

C. The toroidal ITG benchmark test

We choose base case parameters for a closed flux surface edge plasma case to define values for n_e, T_e, T_i, L_n. Dimensional benchmark parameters for case #1 are: $\mu = Z = 1$ (Hydrogen), $n_e = 4.77 \times 10^{12}$ cm^{-3}, $L_n = 1.09$ cm, $T_i = T_e = 17.0$ eV, $L_{Ti} = 0.260$ cm, $\kappa_n = 0.00497$ cm^{-1} (~ 1/R), $B = 1.57 \times 10^4$ G, $\rho_s = 0.0268$ cm (i.e. evaluated where we will apply local theory), $c_s = 4.03 \times 10^6$ cm/s (i.e. evaluated where we will apply local theory). The corresponding dimensionless benchmark parameters are: $k_b\rho_s = 0$ to 1 (will be scanned), $\varepsilon_n = 2\, L_n\, \kappa_n = 0.0108$, $\tau = 1$, $\eta_i = 4.20$. (For these parameters, the Sandberg-dimensionless results can be converted to CGS units by multiplying the dimensionless k_b by 1830 to get n and multiplying the Sandberg γ by 3.69×10^6 to get 1/s.)

For a second case, case #2, which has a broader instability band in k_b, we artificially increase κ_n and hence ε_n by a factor of 10, $\kappa_n = 10*0.00497$ cm^{-1} (~ 1/R), $\varepsilon_n = 0.108$.

Figure 5 shows the comparison of 2DX code results with semi-analytic growth rates for benchmark cases #1 and #2.

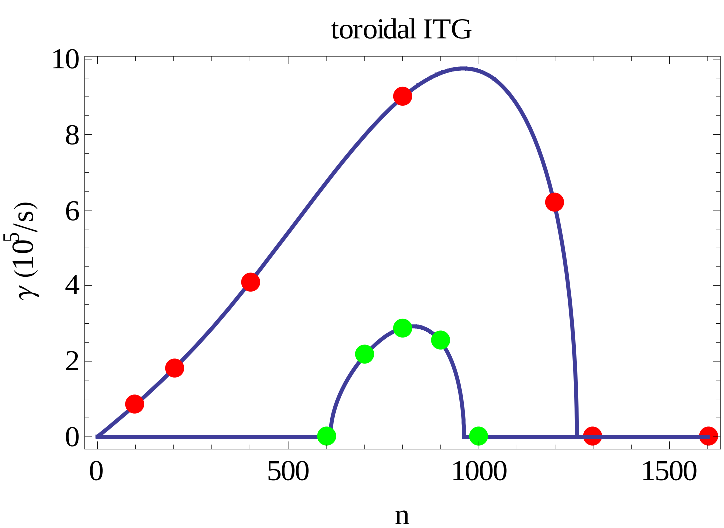

Fig. 1 ITG growth rate for the 2DX benchmark cases #1 (green) and #2 (red). Solid lines are the analytical result, disks are the 2DX results.

n	γ (10^5 s^{-1}) case #1	γ (10^5 s^{-1}) case #2
100	0.000	0.852
200	0.000	1.792
300	0.000	2.870
400	0.000	4.089
500	0.000	5.401
600	0.000	6.729
700	1.111	7.967
800	2.045	8.986
900	1.201	9.629
1000	0.000	9.685
1100	0.000	8.804
1200	0.000	6.073
1300	0.000	0.000

Table 1. Table of semi-analytic growth rates for the toroidal ITG benchmark cases shown in Fig. 1. These are the target results for the benchmark test of the numerical code (2DX).

References

1. see e.g. J. Chen and A.K. Sen, Phys. Rev. Lett. **72**, 3997 (1994) for a brief summary of the slab and toroidal branches.
2. I. Sandberg, Phys. Plasmas **12**, 050701 (2005); and Refs. therein.
3. D. A. Baver, J. R. Myra and M.V. Umansky, Comp. Phys. Comm. **182**, 1610, (2011).
4. http://www.grycap.upv.es/slepc/