Kinetic benchmark of the 2DX code

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1 Introduction

This test was devised to demonstrate and verify the ability of the 2DX eigenvalue code to solve a kinetic resistive ballooning model. Kinetic in this context refers to the inclusion of electron Landau damping into the basic physics model. Since the 2DX code is by nature a fluid code, this represents a signficant extension of the code's capabilities. This extension is achieved through the use of an iterative method which progressively refines approximations to kinetic terms.

Fluid codes can be used to model kinetic effects using gyrofluid or other fluid moment models [1]. This offers a considerable advantage in computational cost compared to a fully kinetic model. However, such models are at best approximations, and in some cases involve non-analytic functions of wavenumber, hence cannot be expressed in sparse matrix form. The iterative method used in this report solves these problems by "tuning" the model equations to maximize accuracy for a specific eigenmode of interest.

This test compares 2DX results to results from a Mathematica test of a spectral kinetic model [2]. The spectral model uses an exact plasma response function, and therefore may be considered as a fully kinetic benchmark case.

2 Description

2.1 Code structure

The 2DX code is a highly flexible eigenvalue solver designed for problems relevant to edge physics in toroidal plasma devices. Its flexibility stems from the use of a specialized input file containing instructions on how to set up a particular set of equations. Because of this, the 2DX code permits model equations to be changed without altering its source code. The drawback to this approach is that any change to the structure file represents a potential source of error, necessitating re-verification. This problem is offset by the fact that the source

code remains unchanged, thus testing one structure file builds confidence in the underlying code that interprets the structure file. Also, structure files can be translated into analytic form, thus allowing the user to verify that the file contains the equations intended.

The structure file contains two main parts: an elements section, which constructs the differential operators and other functions used in a particular set of equations, and a formula section, which assembles these into an actual set of equations. This separation means that elements can be recycled in other structure files. By testing one structure file, one builds confidence in the elements used in that file. The main source of error when switching to a different structure file then is in the formula section, which can be manually verified by translating into analytic form.

Regardless of the content of the structure file, the 2DX code is fundamentally a finite-difference eigenvalue solver. As such, it is subject to the limitations of any code of its type.

2.2 Model equations

For this test we use the following model equations:

$$\gamma \nabla_\perp^2 \delta\phi = \frac{2B}{n} C_r \delta p - \frac{B^2}{b} \partial_\parallel \nabla_\perp^2 \delta A \tag{1}$$

$$\gamma \delta n = -\delta v_E \cdot \nabla n \tag{2}$$

$$-\gamma \nabla_\perp 2\delta A = \nu_e \nabla_\perp^2 \delta A - \mu n \nabla_\parallel \delta\phi \tag{3}$$

$$\delta p = (T_e + T_i)\delta n + n(\delta T_e + \delta T_i) \tag{4}$$

$$C_r = \mathbf{b} \times \kappa \cdot \nabla = -\kappa_g R B_p \partial_x + i(\kappa_n k_b - \kappa_g k_\psi) \tag{5}$$

$$\nabla_\perp^2 = -k_b^2 - \jmath B(k_\psi - i\partial_x R B_p)(1/\jmath B)(k_\psi - i R B_p \partial_x) \tag{6}$$

$$\partial_\parallel Q = B \nabla_\parallel (Q/B) \tag{7}$$

$$\nabla_\parallel = \jmath \partial_y \tag{8}$$

$$\delta v_E \cdot \nabla Q = -i \frac{k_z (R B_p \partial_x Q)}{B} \delta\Phi \tag{9}$$

$$\nu_e = .51 \nu_r n / T_e^{3/2} \tag{10}$$

In this notation, κ_g is geodesic curvature, κ_n is normal curvature, k_b is binormal wavenumber, k_ψ is radial wavenumber. RB_p is poloidal flux density, as polodial flux is used as a radial coordinate, $\jmath$ is the inverse Jacobian $1/JB$ which is used to define the poloidal coordinate, and Q is any quantity. The above equations are normalized to Bohm units, i.e. all distances are in units of ρ_s and all time scales are in units of ω_{ci}^{-1}.

Kinetic effects are modeled by defining conductivity in terms of the plasma response function:

$$\nu_e = \mu/\sigma \tag{11}$$

$$\sigma = \frac{i\Omega_e}{\omega + i\nu_0}\zeta^2 Z'(\zeta) \tag{12}$$

$$\zeta = \frac{\omega + i\nu_0}{|k_\parallel|\sqrt{2}v_{the}} \tag{13}$$

where ν_0 is the physical resistivity, as opposed to ν_e which in these equations is used to model the effective resistivity.

Since the definition of ζ contains a non-analytic function of the wavenumber, it cannot be expressed in terms of spatial finite difference operators. In order to solve this problem, an iterative approach is used.

2.3 Iterative method

In order to calculate the plasma response function without using non-analytic functions of wavenumber, a polynomial approximation is used. In order to make this approximation as accurate as possible, the coefficients of this polynomial are iteratively updated to produce the most accurate possible fit at a particular wavenumber and growth rate. The growth rate for which the fit is to be optimized can be calculated using the eigenvalue of a previous iteration of the code, whereas the wavenumber can be calculated by doing simple numerical analysis of the eigenvector.

In this approach, conductivity is first represented as a polynomial in k:

$$\sigma = \frac{\mu}{\nu_e(a + bk^2)} \tag{14}$$

This formula can be represented in operator form by making minor modifications to Eq. 3. This yields:

$$-\gamma\nabla_\perp 2\delta A = \nu_e a\nabla_\perp^2\delta A - \nu_e b\nabla_\perp^2\nabla_\parallel^2\delta A - \mu n\nabla_\parallel\delta\phi \tag{15}$$

The coefficients a and b are then calculated using the plasma response function:

$$a = \frac{\beta k^3\sigma_0}{2\alpha^2} \tag{16}$$

$$b = \frac{\sigma_0(2\alpha - \beta k)}{2\alpha^2} \tag{17}$$

$$\alpha = \frac{\Omega_e}{\nu + \gamma}\zeta^2 Z'(\zeta) \tag{18}$$

$$\beta = -\frac{\Omega_e}{\nu + \gamma}\zeta^3 Z''(\zeta) \tag{19}$$

The wavenumber $k_\parallel$ can be extracted from the eigenvector by taking an average derivative:

$$k_\parallel = \sqrt{\frac{\sum_i \jmath_i^2 |\psi_i - \psi_{i-1}|^2}{dy^2 \sum_i |\psi_i|^2}} \tag{20}$$

2.4 Boundary conditions

This test case uses phase-shift periodic boundary conditions in the parallel direction, and zero-derivative boundary conditions in the radial direction. The phase shift in the parallel direction is given by:

$$\delta Q(y = 0) = \delta Q(y = 2\pi) e^{-i2\pi nq} \tag{21}$$

This ensures toroidal and poloidal periodicity in the field-line following coordinate.

2.5 Profile setup

The formulas in Eq. 1-3 are normalized to Bohm units. Values are converted by dividing input distances by ρ_s, and input magnetic fields are in Tesla. Output eigenvalues are multiplied by ω_{ci}. Resistivity is given by the formula:

$$\nu_r = \frac{\mu}{.51\sigma_0} \tag{22}$$

where

$$\sigma_0 = 1.96 \frac{\omega_{ce}}{\nu_{ei}} \tag{23}$$

The geometry used is an idealized toroidal annulus with major radius R, minor radius a, and thickness δa. The density profile is exponential with scale length L_n, and temperature profiles are flat. Curvature is assumed, and is given by:

$$\kappa_n = \frac{cos(y)}{R} \tag{24}$$

The function q may be sheared, but shear is set to zero for the test case given. The value of this constant q is given in Sec. 3.

Parallel derivatives are calculated using the Jacobian factor $\jmath = 1/qR$. Toroidal mode number is calculated by $n = k_z a/q_0$.

3 Numerical results

The code and iterative method were tested by sweeping the variable T_e from 10 to 200. Ten iterations were used for each value for the kinetic method. The results from the kinetic iterative method were compared to results from the fluid model. In addition, a spectral calculation was used to determine the solution for the kinetic case using the full Z function rather than an approximation [2]. The other parameters used were:

$a = .75$ cm
$\delta a = .3$ cm
$R = 207.5$ cm
$L_n = 4$ cm
$Z_{eff} = 1$
$B = 3$ T
$n_e = 10^{13} cm^{-3}$
$m_i/m_p = 2$
$\mu = 3674.32$
$T_i = 0$
$ln\Lambda = 24 - Log(n_e/T_e)$

The results of this test are shown in Fig. 1. In addition, a table of the raw eigenvalue data is shown in table 2. This test compares the 2DX kinetic iterative method (yellow diamonds), the 2DX fluid model (green triangles), the spectral full kinetic model (blue circles), and the spectral fluid model (red squares). As can be seen from this data, there are some slight discrepancies between 2DX and the spectral method even in the fluid case. This indicates the relative limitations of comparing a spectral to a spatial model; in particular, the spatial model is subject to numerical dispersion due to finite resolution. The discrepancy between the iterative and spectral kinetic methods is slightly larger, but it is still small enough for the method to be useful. This discrepancy can be explained because the eigenmode is not a pure sinusoidal function, so it does not have a single wavenumber. The iterative method is designed to get as good as possible a fit to a full plasma response function over a range of wavenumbers, but there are limits to how good a fit is possible.

References

[1] G. W. Hammett, F. W. Perkins, Phys. Rev. Lett. **25**, 3019 (1990).

[2] 2DX Phase I final report

nx	2	ny	64
dx	$.3/\rho_s$	dy	.097331
γ	$\gamma(s^{-1})/1.437 \times 10^8$	n	5
μ	3674.32	ν_r	$\nu_{ei}\mu/1.96\omega_{ce}$
q	3.3	k_z	$22\rho_s$
$\jmath$	$\rho_s 668.25$	k_ψ	0
κ_n	$\rho_s/202.5$ cos(y)	κ_g	0
B	3	RB_p	1

Table 1: Non-dimensional profile functions and parameters used in the resistive ballooning test case, as a function of the dimensional input $k_z(cm^{-1})$.

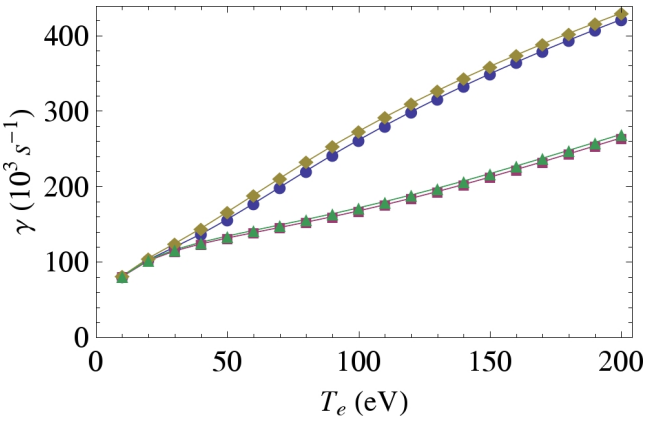

Figure 1: Growth rate vs. electron temperature for fluid and kinetic resistive ballooning models. Yellow diamonds are 2DX results using the iterative kinetic model. Green triangles are 2DX results using the fluid model. Blue circles are solutions to the kinetic model using a spectral method, and red squares are solutions to a fluid model using a spectral method.

T_e	γ (2DX kinetic)	γ (spectral kinetic)	γ (2DX fluid)	γ (spectral fluid)
10	81803	81079	81545	81007
20	105501	103325	103223	102137
30	124705	120139	116570	114943
40	144832	137440	126410	124286
50	166502	156785	134605	132036
60	188948	177778	142069	139109
70	211303	199440	149304	146005
80	232992	220954	156606	153013
90	253750	241840	164149	160306
100	273509	261883	172035	167982
110	292299	281027	180317	176090
120	310192	299295	189011	184644
130	327273	316746	198107	193630
140	343622	333449	207574	203015
150	359316	349472	217368	212751
160	374418	364882	227439	222781
170	388989	379736	237729	233047
180	403077	394085	248184	243489
190	416725	407975	258749	254053
200	429972	421445	269379	264688

Table 2: Growth rate vs. T_e for the kinetic and fluid resistive ballooning models.